On Recent Advances of the 3D Navier-Stokes and Euler Equations

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   - Classical results of local well-posedness

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   - The effect of boundary conditions
   - Deos the advection term deplete singularity?

Euler equations

We consider the Euler Equations of inviscid incompressible fluid in $\Omega = \mathbb{R}^3$, the whole space, or $\Omega = (\mathbb{R}/\mathbb{Z})^3$, the three dimensional torus.

$$\begin{aligned} u_t + (u \cdot \nabla)u + \nabla p &= 0, \quad x \in \Omega, \quad t > 0 \\ \nabla \cdot u &= 0, \\ u(x,0) &= u_0(x). \end{aligned}$$

where the velocity field $u = (u_1, u_2, u_3)$ and pressure p are unknowns.

Vorticity formulation

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the velocity can be recovered from the vorticity via the Biot-Savart law in $\mathbb{R}^3$

$$u(x,t) = \frac{1}{4\pi} \int_{\mathbb{R}^3} \frac{(x-y) \wedge \omega(y)}{|x-y|^3} dy\,.$$

Classical well-posedness results

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- Question: Does there exist a regular solution (say in $C^{1,\alpha}$) of the $3d$ Euler equations that becomes singular in a finite time (blows up problem)?

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- Elgindi [2020] For $0 < \alpha \ll 1$ the $3d$ axi-symmetric Euler equations develop finite-time singularity in the space $C^{1,\alpha}$. These solutions are in the whole space and have infinite energy.

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- Chen-Hou [2020] Similar result the $3d$ axi-symmetric Euler equations in a cylinder, i.e., with physical boundaries.

A trivial blowup criterion

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$$\xi = \frac{\omega}{|\omega|}$$

Consider the Euler equations in a smooth bounded domain Ω with boundary condition $u \cdot \vec{n}|_{\partial\Omega} = 0$.

New Blow up Criterion in Bounded Domain

Consider the Euler equations in a smooth bounded domain Ω with boundary condition $u \cdot \vec{n}|_{\partial\Omega} = 0$.
Let θ, q and B be defined as follows:

$$\frac{\partial \theta}{\partial t} + u \cdot \nabla \theta = 0, \quad q = \omega \cdot \nabla \theta, \quad \text{and} \quad B = \nabla q \times \nabla \theta.$$

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Theorem (Gibbon-Titi [J. Nonlinear Science 2013])

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Lack of uniqueness of weak solutions/Convex integration

Theorem DeLellis - Szekelyhidi

There exist a set of initial data $u_0 \in L^2(\Omega)$ *(not explicitly constructed) for which the Cauchy problem has, for the same initial data, an infinite family of weak solutions of the three-dimensional Euler equations: a residual set in the space* $C(\mathbb{R}_t; L^2_{\text{weak}}(\Omega))$.

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- Remark: Earlier results were established by Shnirelman and by V. Sheffer.

Existence of Weak solutions to the Cauchy Problem

Theorem Wiedemann (2011).

There exists a family (non-uniqueness) of weak solutions to the Cauchy problem of the 3D Euler.

Ruling Out Principle of Wild Weak Solutions in the Absence of Physical Boundaries

Ruling Out Principle

Any wild weak solution of Euler equations, in domains without physical boundaries, that cannot be achieved as a vanishing viscosity limit of a Leray-Hopf weak solutions of the Navier-Stokes equations should be ruled out.

Does 2D Flow Remain 2D?

Theorem Bardos, Lopes-Filho,Nussenzveig-Lopes, Niu, Titi –[SIAM, Jour. Math. Analysis (2013)].

Let u_0 *be a function of* (x, y) *only,*

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Let u_0 be a function of (x, y), then there is a unique corresponding Leray-Hopf weak solution of the 3D Navier-Stokes. Moreover, this solution remains a function of only (x, y).

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Let u_0 be a function of (x, y), then there is a unique corresponding Leray-Hopf weak solution of the 3D Navier-Stokes. Moreover, this solution remains a function of only (x, y). Similar result for axi-symmetric initial data, or helical initial data.

Ruling Out Symmetry Breaking Solutions

Ruling out principle: all the wild weak solutions of Euler equations that do not obey the two-dimensional symmetry of the initial data should be **ruled out**. Because they cannot be obtained as vanishing viscosity limit of Leray-Hopf weak solutions of the Navier-Stokes equations.

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- Consider the 3D hyper-viscous Navier-Stokes equations:

$$\begin{aligned} u_t + \nu(-\Delta)^{\theta} u + (u \cdot \nabla) u + \nabla p &= 0, \quad x \in \Omega, \;\; t > 0 \\ \nabla \cdot u &= 0. \end{aligned}$$

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- Luo-Titi (2018) Constructed, for all $\theta < 5/4$, non-unique weak solutions of the 3D hyper-viscous Navier-Stokes equations, following and extending the Buckmaster-Vicol procedure.
- Sharpness of convex integration machinery and Lions.

The Positive Part of the Onsager Conjecture

Theorem Constantin-E-Titi (1994)

Let u be a weak solution of the Euler equations, such that $u \in L^\infty((0,T); L^2(\mathbb{R}^3)) \cap L^3((0,T); C^{0,\alpha}(\mathbb{R}^3))$, with $\alpha > 1/3$, then u conserves the energy.

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Theorem Bardos-Titi (2017)

Let Ω be a bounded smooth domain, and let u be a weak solution of the Euler equations, such that $u \cdot \vec{n}|_{\partial\Omega} = 0$ and $u \in L^\infty((0,T); L^2(\Omega)) \cap L^3((0,T); C^{0,\alpha}(\overline{\Omega}))$, with $\alpha > 1/3$, then u conserves the energy.

Remarks on the Positive Part of the Onsager Conjecture

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- Extensions to other physical systems, cf. Drivas-Eyink, Feireisl-Gwiazda-Świerczewska–Gwiazda-Wiedemann, Gwiazda-Michálek-Świerczewska–Gwiazda, Yu, etc...

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Shear-flow

We consider in the whole space $\mathbb{R}^3$ or on the periodic box $(\mathbb{R}/\mathbb{Z})^3$:

The shear flow

$$u(x,t) = (u_1(x_2), 0, u_3(x_1 - tu_1(x_2)).$$

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- Moreover, in the case of the periodic box $(\mathbb{R}/\mathbb{Z})^3$ the above shear flow conserves the energy.

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Idea of the proof

$$\partial_{x_2} u_3(x_1 - tu_1(x_2)) = -t\partial_{x_2} u_1(x_2)\partial_{x_1} u_3(x_1 - tu_1(x_2))$$

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Theorem Bardos-Titi

(i) Let $u_1, u_3 \in L^2_{\text{loc}}(\mathbb{R})$ *then the shear flow*

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Shear flow is a weak solution that conserves energy

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Ill-posedness of Euler equations in $C^{0,\alpha}$

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(ii) For $u_1(x), u_3(x) \in C^{0,\alpha}$, with $\alpha \in (0,1)$, the above shear flow solution of the three-dimensional Euler equations is always in C^{0,α^2}.

(iii) There exist shear flow solutions of the above form which, for $t = 0$, belong to $C^{0,\alpha}$, for some $\alpha \in (0,1)$, and for $t \neq 0$, they do not belong to $C^{0,\beta}$ for any $\beta > \alpha^2$.

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Other spaces and optimal spaces

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$$C^{1,\alpha} = B^{1+\alpha}_{\infty,\infty} \subset B^{1}_{\infty,1} \subset C^{1} \subset F^{1}_{\infty,2} \subset B^{1}_{\infty,\infty} \subset B^{\alpha}_{\infty,\infty} = C^{0,\alpha} .$$

Theorem *The $3d$ Euler equation is well posed in $B^1_{\infty,1}$* (Pak and Park). It is not well posed in $B^1_{\infty,\infty}$ or in the Triebel-Lizorkin space $\subset F^1_{\infty,2}$

Theorem Bourgain-Li & Elgindi-Masmoudi C^m Ill-posendness

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Analyticity of Solutions

It is clear that if u_1, u_3 are analytic functions then the shear-flow solution is also analytic for a short while.

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to provide an explicit example for an analytic solutions whose radius of analyticity is shrinking with the rate $\frac{1}{t}$.

Vanishing viscosity a ruling out principle

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Let $v_0(x) = (v_1(x_2), 0, v_3(x_1, x_2))$, *where we assume* $v_1 \in L^2(\mathbb{T})$ *and* $v_3 \in L^2(\mathbb{T}^2)$.

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corresponding to v_0, as $\nu \to 0$.

Shear flow with vorticity interface

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$$\begin{aligned} \Sigma(t) = \{(x_1, x_2, x_3) | \, x_2 = \xi_2\} \cup \\ \{(x_1, x_2, x_3) | \, x_1 = \xi_1 + t\alpha_1, \, x_2 \leq \xi_2\} \\ \cup \{(x_1, x_2, x_3) | \, x_1 = \xi_1 + t\beta_1, \, x_2 \geq \xi_2\} \, . \end{aligned}$$

- In the problem under consideration the vorticity is concentrated on an orientable curve, in $2d$, or a surface in $3d$.

$2d$ and $3d$ Kelvin-Helmholtz problem - a comparison

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- In the second example the function $x_2 \mapsto u_1(x_2)$ does not seem to require more than C^1 regularity in order to maintain this regularity. For the two-dimensional Kelvin-Helmholtz (Birkhoff-Rott) such property is not possible, while it might be possible in the three-dimensional case.

Let $\Gamma(t)$ be the vortex-sheet, the interface where the vorticity is concentrated. The velocity outside interface $\Gamma(t)$ is given by:

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- In the three-dimensional case:

$$u(x,t) = \\ -\frac{1}{4\pi} \int \frac{x - r(t,\lambda',\mu')}{|x - r(t,\lambda',\mu')|^3} \wedge \tilde{\omega}(t, r(t,\lambda',\mu')) |\partial_\lambda (r(t,\lambda')) \wedge \partial_\mu (r(t,\lambda'))| d\lambda' d\mu'$$

Vorticity density

When the point x converges to a point $r \in \Gamma(t)$ the velocity $u(x,t)$ converges to two different values $u_{\pm}(r)$ which satisfy:

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$$
\begin{aligned}
&\partial_t y - v_2 = -(v_1 \partial_x y)\,,\\
&\partial_t \tilde{\omega} + \partial_x (v_1 \Omega_0) = -\epsilon \partial_x (v_1 \tilde{\omega})\,,\\
&v_1(x,t) = -\frac{1}{2\pi} P.V. \int \frac{y(x,t) - y(x',t)}{(x-x')^2 + \epsilon^2 (y(x,t) - y(x',t))^2} (\Omega_0 + \epsilon \tilde{\omega}) dx'\,,\\
&v_2(x,t) = \frac{1}{2\pi} P.V. \int \frac{x - x'}{(x-x')^2 + \epsilon^2 (y(x,t) - y(x',t))^2} (\Omega_0 + \epsilon \tilde{\omega}) dx'\,.
\end{aligned}
$$

This system describes perturbations in $\mathbb{R}^2$ about the stationary solution

$$
y(x,0) = 0\,, u_- = \frac{\Omega_0}{2}\,, u_+ = -\frac{\Omega_0}{2}\,.
$$

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Local ellipticity continue ...

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This in turn leads to the introduction of the operators (Hilbert transform):

$$Hf(x) = \frac{1}{\pi}\int \frac{1}{x-x'}f(x')dx' = F^{-1}(-i\mathrm{sgn}(\xi)\hat{f}(\xi))$$
$$|D|f(x) = \frac{1}{\pi}P.V.\int \frac{f(x)-f(x')}{(x-x')^2} = \partial_x(Hf(x)) = F^{-1}(|\xi|)\hat{f}(\xi))\,.$$

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In the right hand side F and G are first order operators. Eventually with the introduction of the "Laplacian" one has:

$$\partial_{tt}(y_x) + \Omega_0^2 \partial_{xx}(y_x) = \epsilon(\partial_t(F(y_x, \omega)_x) + |D|(\epsilon G(y_x, \omega)_x)\,,$$
$$\partial_{tt}(\omega) + \Omega_0^2 \partial_{xx}(\omega) = \epsilon(|D|(F(y_x, \omega)_x) + \partial_t(\epsilon G(y_x, \omega)_x)\,.$$

What is the situation in the three-dimensional Kelvin-Helmholtz?

Repeat the previous analysis for

$$\Gamma(t) = \{x_3 = \epsilon x(x_1, x_2, t)\},$$

a *small* perturbation about the stationary flat state $x_3 = 0, \tilde{\omega}^0(x_1, x_2) = (\tilde{\omega}_1^0, \tilde{\omega}_2^0, 0)$.

Local analysis for $3d$ Kelvin-Helmholtz

Leading part of the perturbed equation $\partial_t \begin{pmatrix} \hat{x_3} \\ \hat{\omega_1} \\ \hat{\omega_2} \\ \hat{\omega_3} \end{pmatrix} = \mathcal{A} \begin{pmatrix} \hat{x_3} \\ \hat{\omega_1} \\ \hat{\omega_2} \\ \hat{\omega_3} \end{pmatrix}$

with $k = |k|(\cos\theta, \sin\theta)$ and

$$\mathcal{A} = \begin{pmatrix} 0 & \frac{i}{2}\sin\theta & -\frac{i}{2}\cos\theta & 0 \\ -\frac{i}{2}|k|^2|\omega^0|^2\sin\theta & 0 & 0 & \frac{1}{2}(k\cdot\omega^0)\sin\theta \\ \frac{i}{2}|k|^2|\omega^0|^2\cos\theta & 0 & 0 & -\frac{1}{2}(k\cdot\omega^0)\cos\theta \\ 0 & -\frac{1}{2}(k\cdot\omega^0)\sin\theta & \frac{1}{2}(k\cdot\omega^0)\cos\theta & 0 \end{pmatrix}$$

The eigenvalues of the matrix $\mathcal{A}$ are

$$\{0, 0, -\frac{1}{2}|k \wedge \tilde{\omega}^0|, \frac{1}{2}|k \wedge \tilde{\omega}^0|\}\,.$$

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Therefore, the first order pseudo-differential operator

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is no longer elliptic. Which was the basic reason of why a smooth (with limited regularity) singular support may persist without being in fact analytic.

Local analysis of Kelvin-Helmholtz continue ...

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That is the three-dimensional Kelvin-Helmholtz (Birkhoff-Rott) problem is more stable than the two-dimensional one!!!

Numerical investigation of blow up - Hyperviscosity

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Numerical investigation of blow up - Hyperviscosity

It is well-known that the viscous Burgers equation (viscous Hamilton-Jacobi):

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Does not develop singularity in finite time. How about the hyper-viscous Hamilton-Jacobi:

$$\frac{\partial u}{\partial t} + \nu \Delta^{2m} u = |\nabla u|^2?$$

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Answer [Larios-Titi]:
NO in periodic boundary conditions.

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Answer [Larios-Titi]:
NO in periodic boundary conditions.
YES with Dirichlet boundary conditions.

Dropping Advection Causes Blow-up for 3D Euler Equations

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This equation blows up in finite time Constantin [Commun. Math. Phys. 1986].

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Recent Computations of Luo and Hou

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They considered special class of solutions that vanish at the axis.

They show, computationally, that certain solutions blow-up at the boundary.

Euler equations:

$$\begin{aligned} u_t + (u \cdot \nabla) u + \nabla p &= 0, \quad x \in \Omega, \quad t > 0 \\ \nabla \cdot u &= 0, \\ u(x,0) &= u_0(x). \end{aligned}$$

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Inviscid Regulations of the Euler Equations

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Euler-Voigt equations:

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Blowup criterion [Larios-Titi]:

$$\liminf_{\alpha \to 0} \sup_{0 \le t \le T} \alpha^2 \|\nabla u(t)\|_{L^2}^2 > 0.$$

Computation Study Implementing the Euler-Voigt Criterion

Computational study by Larios-Petersen-Titi-Wingate [Theor. Comput. Fluid Dyn. (2018)], implementing the Euler-Voigt regularization:

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Computation Study Implementing the Euler-Voigt Criterion

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and the corresponding **Blowup Criterion**

$$\liminf_{\alpha \to 0} \sup_{0 \le t \le T} \alpha^2 \|\nabla u(t)\|_{L^2}^2 > 0,$$

show

- Finite-time singularity of the $3d$ Euler equations.
- Finite-time singularity of the inviscid Burgers equations (regularized by the Benjamin-Bona-Mahony equation).

THANK YOU FOR YOUR ATTENTION!